NOTE ON TORSIONAL VIBRATIONS OF NON-HOMOGENEOUS SPHERICAL AND CYLINDRICAL SHELLS

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ABSTRACT. This paper deals with the torsional vibration of non-homogeneous thick spherical and cylindrical shells. Non-homogeniety arises due to variable density ρ and rigidity Modulus μ. The laws of non-homogeniety are $\rho = \rho_0 r^n$ and $\mu = \mu_0 r^n$. The frequency equations are given and numerical evaluation of roots are presented for some particular cases.

INTRODUCTION

In this note some problems of elastic vibration of non-uniform and non-homogeneous thick shells are investigated. In the first case we shall consider the torsional vibration of a spherical shell in which $r = a$ and $r = b$ are internal and external radii of the shell respectively. Both inner and outer boundaries are free. In the second case we shall consider torsional vibration of a cylindrical shell with c and d as inner and outer radii. We have taken same power law variation of elastic constant and density of the material composing thick shells.

Case 1. If we suppose that the components of displacement u_r and u_θ are zero and that the azimuthal component $w(= u_\phi)$ is independent of ϕ we have components of stress in spherical polar co-ordinates as

$$\left.\begin{aligned} \Delta = 0, \quad \widehat{rr} = \widehat{\theta\theta} = \widehat{\phi\phi} = \widehat{r\theta} = 0 \quad \widehat{r\phi} = \mu\left[\frac{\partial\omega}{\partial r} - \frac{\omega}{r}\right] \\ \widehat{\theta\phi} = \frac{\mu}{r}\left[\frac{\partial\omega}{\partial\theta} - \omega\cot\theta\right] \end{aligned}\right\} \quad \ldots (1)$$

The stress equation of motion satisfied by w is

$$\frac{\partial}{\partial r}\widehat{r\phi} + \frac{1}{r}\frac{\partial}{\partial\theta}\widehat{\theta\phi} + \frac{3}{r}\widehat{r\phi} + 2\frac{\widehat{\theta\phi}}{r}\cot\theta = \rho\frac{\partial^2\omega}{\partial t^2} \quad \ldots (2)$$

Let $\qquad \mu = \mu_0 r^n \quad$ and $\quad \rho = \rho_0 r^n$

Then $$\widehat{r\phi} = \mu_0 r^n\left[\frac{\partial\omega}{\partial r} - \frac{\omega}{r}\right], \quad \widehat{\theta\phi} = \mu_0 r^{n-1}\left[\frac{\partial\omega}{\partial\theta} - \omega\cot\theta\right] \quad \ldots (3)$$

Substituting (3) in theequation (2) we get

$$\left[r^n \frac{\partial^2\omega}{\partial r^2} + (n+2)r^{n-1}\frac{\partial\omega}{\partial r} - (n+2)r^{n-2}\omega\right]$$

$$+r^{n-2}\left[\frac{\partial^2\omega}{\partial\theta^2} + \cot\theta\frac{\partial\omega}{\partial\theta} + (1-\cot^2\theta)\omega\right] = \frac{\rho_0 r^n}{\mu_0}\frac{\partial^2\omega}{\partial t^2} \qquad \ldots (4)$$

For rotatory vibration of the shell, we assume

$$u_r = 0 = u_\theta, \quad u_\varphi = \omega = f(r)\sin\theta e^{ipt}$$

Then equation (4) reduces to

$$r^2 f''(r) + (n+2)rf'(r) + \{\lambda^2 r^2 - (n+2)\}f(r) = 0 \qquad \ldots (5)$$

where $$\lambda^2 = \frac{\rho_0 p^2}{\mu_0}$$

The solution of equation (5) is

$$f(r) = r^{-\frac{n+1}{2}}\left[AJ_{\frac{n+3}{2}}(\lambda r) + BY_{\frac{n+3}{2}}(\lambda r)\right] \qquad \ldots (6)$$

So $$\omega = r^{-\frac{n+1}{2}}\left[AJ_{\frac{n+3}{2}}(\lambda r) + BY_{\frac{n+3}{2}}(\lambda r)\right]\sin\theta e^{ipt}$$

Boundary condition : We assume that $\widehat{\phi r} = 0$ when $r = a$ and $r = b$ ... (7)

Now $$\widehat{\phi r} = \mu_0 r^n\left[\frac{\partial\omega}{\partial r} - \frac{\omega}{r}\right]$$

$$= -\mu_0\lambda r^{(3n+5)/2}\left[AJ_{\frac{n+5}{2}}(\lambda r) + BY_{\frac{n+5}{2}}(\lambda r)\right]\sin\theta e^{ipt} \qquad \ldots (8)$$

From (7) and (8) we have

$$\left.\begin{aligned} AJ_{\frac{n+5}{2}}(\lambda a) + BY_{\frac{n+5}{2}}(\lambda a) = 0 \\ AJ_{\frac{n+5}{2}}(\lambda b) + BY_{\frac{n+5}{2}}(\lambda b) = 0 \end{aligned}\right\} \qquad \ldots (9)$$

Eliminating A and B from from (9) we have

$$J_{\frac{n+5}{2}}(\lambda a)\, Y_{\frac{n+5}{2}}(\lambda b) - J_{\frac{n+5}{2}}(\lambda b)\, Y_{\frac{n+5}{2}}(\lambda a) = 0 \qquad \ldots (10)$$

Equation (10) gives the frequency equation of torsional or rotatory vibration of the spherical shell.

Case 2. In case of thick cylindrical shell, taking the axis of the cylindrical shell as the axis of z and assuming $u = \omega = 0$ and v is independent of θ, we have stress components in cylindrical co-ordinates as

$$\widehat{rr} = \widehat{\theta\theta} = \widehat{zz} = \widehat{rz} = 0, \quad \widehat{\theta z} = \mu \frac{\partial v}{\partial z}, \quad \widehat{r\theta} = \mu . r \frac{\partial}{\partial r}\left(\frac{v}{r}\right) \qquad \ldots \quad (1)$$

Equations of motion in terms of displacements are

$$\left.\begin{aligned} &\frac{\partial}{\partial r}\widehat{rr} + \frac{1}{r}\frac{\partial}{\partial \theta}\widehat{r\theta} + \frac{1}{r}(\widehat{rr} - \widehat{\theta\theta}) = \rho \ddot{u} \\ &\frac{\partial}{\partial r}\widehat{r\theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\widehat{\theta\theta} + \frac{\partial}{\partial z}\widehat{\theta z} + \frac{2}{r}\widehat{r\theta} = \rho \ddot{v} \\ &\frac{\partial}{\partial r}\widehat{rz} + \frac{1}{r}\frac{\partial}{\partial \theta}\widehat{\theta z} + \frac{\partial}{\partial z}\widehat{zz} + \frac{1}{r}\widehat{rz} = \rho \ddot{w} \end{aligned}\right\} \qquad \ldots \quad (2)$$

If we assume $\mu = \mu_0 r^n$ and $\rho = \rho_0 r^n$ where μ_0 and ρ_0 are constants and substitute (1) in (2) we find that the first and third equationr of (2) are identically satisfied and the second equation gives

$$r^n \frac{\partial^2 v}{\partial r^2} + (n+1) r^{n-1} \frac{\partial v}{\partial r} + r^n \frac{\partial^2 v}{\partial z^2} - (n+1) r^{n-2} v = \frac{\rho_0}{\mu_0} r^n \frac{\partial^2 v}{\partial t^2} \qquad \ldots \quad (3)$$

Assuming $v = C \cos \gamma z \, V(r) e^{ipt}$ the equation (3) reduces to

$$r^2 \frac{d^2 V}{dr^2} + (n+1) r \frac{dV}{dr} + \left(\left[\frac{\rho_0 p^2}{\mu_0} - \gamma^2 \right] r^2 - (n+1) \right) V = 0 \qquad (4)$$

Solution of equation (4) is

$$V(r) = r^{-\frac{n}{2}} \left[A J_{\frac{n}{2}+1}(\lambda r) + B Y_{\frac{n}{2}+1}(\lambda r) \right] \qquad \ldots \quad (5)$$

where
$$\lambda^2 = \frac{\rho_0 p^2}{\mu_0} - \gamma^2$$

Therefore
$$v = C \cos \gamma z r^{-\frac{n}{2}} \left[A J_{\frac{n}{2}+1}(\lambda r) + B_{\frac{n}{2}+1}(\lambda r) \right] e^{ipt} \qquad \ldots \quad (6)$$

The boundary conditions are

$$\left.\begin{aligned} \widehat{\theta z} &= 0 \quad \text{when } z = 0 \\ &= 0 \quad \text{when } Z = L, L \text{ being length of the cylinder} \\ \widehat{r\theta} &= 0 \quad \text{when } r = c \\ &= 0 \quad \text{when } r = d \end{aligned}\right\} \qquad \ldots \quad (7)$$

First condition of (7) is satisfied. From second condition of (7) we have

$$\gamma = \frac{k\pi}{L} \quad \text{where } k \text{ is any integer} \qquad \dots (8)$$

Now $$\widehat{r\theta} = -\mu_0 \lambda r^{\frac{3n+2}{2}} \left[A J_{\frac{n}{2}+2}(\lambda r) + B Y_{\frac{n}{2}+2}(\lambda r) \right] \cos \gamma z e^{ipt} \qquad \dots (9)$$

Third and fourth conditions (7) and (9) give

$$\left. \begin{aligned} A J_{\frac{n}{2}+2}(\lambda c) + B Y_{\frac{n}{2}+2}(\lambda c) &= 0 \\ A J_{\frac{n}{2}+2}(\lambda d) + B Y_{\frac{n}{2}+2}(\lambda d) &= 0 \end{aligned} \right\} \qquad \dots (10)$$

Eliminating A and B from (10) we have

$$J_{\frac{n}{2}+2}(\lambda c) Y_{\frac{n}{2}+2}(\lambda d) - J_{\frac{n}{2}+2}(\lambda d) Y_{\frac{n}{2}+2}(\lambda c) = 0 \qquad \dots (11)$$

This equation gives the torsional vibration of the cylindrical shell. This equation (11) and equation (10) of case 1 are of same form.

Putting $\lambda c = \bar{\omega}$ and $\lambda d = \eta \bar{\omega}$, $\frac{n}{2}+2 = m$

so that $$\eta = \frac{d}{c}$$

(11) can be written as

$$\frac{Y_m(\eta \bar{\omega})}{J_m(\eta \bar{\omega})} = \frac{Y_m(\bar{\omega})}{J_m(\bar{\omega})} \qquad \dots (12)$$

It is known (from Gray and Mathews, 1931, P261) that the s-th root in order of magnitude, of equation

$$\frac{Y_m(\bar{\omega})}{J_m(\bar{\omega})} - \frac{Y_m(\eta \bar{\omega})}{J_m(\eta \bar{\omega})} = 0, \quad \eta > 1$$

is $$\bar{\omega}^{(s)} = \partial + \frac{\alpha}{\partial} + \frac{\beta - \alpha^2}{\partial^3} + \frac{\theta' - 4\alpha\beta + 2\alpha^3}{\partial^5} + \dots$$

where $$\partial = \frac{s\pi}{\eta - 1}, \qquad \alpha = \frac{4m^2 - 1}{8\eta}$$

$$\beta = \frac{4(4m^2-1)(4m^2-25)(\eta^3-1)}{3(8\eta)^3(\eta-1)}$$

$$\theta' = \frac{32(4m^2-1)(16m^4-456m^2+1073)(\eta^5-1)}{5(8\eta)^5(\eta-1)}$$

Roots of the equation (12) have been calculated for $n = 2$ and for different values of η i.e. for different values of the ratio a/c and are given in the following Table

TABLE

	$\frac{c}{d}$	.25	.5	.75
	$\eta = \frac{d}{c}$	4	2	$\frac{4}{3}$
$\tilde{\omega}^{(1)}$		-0 .867	3 .708	9 .754
$\tilde{\omega}^{(2)}$		2 .453	6 .616	19 .025
$\tilde{\omega}^{(3)}$		3 .453	9 .654	28 .394

REFERENCES

Gray and Mathews, 1931, *A Treatise on Bessel functions*, London, pp. 261.
Love, A. E. H., 1920, *A treatise on the mathematical theory of Elasticity*. Oxford.